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Analysis of localization in frame members with plastic hinges

María Eugenia Marante [a,b,*], Ricardo Picón [a,b], Julio Flórez-López [a]

[a] Department of Structural Engineering, Faculated de Ingenieria, University of Los Andes, Mérida 5101, Venezuela
[b] Department of Structural Engineering, Lisandro Alvarado University, Barquisimeto, Venezuela

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Abstract

This paper proposes a general criterion of localization for frame members. A frame member is assumed to be the assemblage of an elastic beam-column and two plastic hinges at the ends of the member. Each plastic hinge presents plastic softening behavior under bending and axial forces. In this case, localization refers to the concentration of plastic rotation within a plastic hinge while the other one unloads. The localization criterion is obtained from the analysis of the problem that defines the plastic multipliers of the hinges as a function of the member displacements rate. Then, a simple uniqueness condition is derived for this problem. If the uniqueness condition is not verified, bifurcated solutions with localization are possible, hence the localization criterion. This criterion is applied in three particular cases for which analytical solutions are obtained and discussed. Finally the numerical implications of the localization criterion are discussed.

Keywords: Localization; Plastic hinges; Softening; Framed structures

1. Introduction

Reinforced concrete or steel framed structures can show significant softening if they are subjected to overloads. In the former case, softening is the result of concrete cracking while in the latter this effect is mainly due to local buckling. These degrading phenomena: extensive plasticity, cracking and local buckling can be lumped at plastic hinges (see for instance, Inglessis et al., 2002; Marante and Flórez-López, 2003).

In framed structures, as well as in continuum media, softening is associated to the phenomenon of localization. Localization in framed structures takes the form of transfer of energy dissipation amongst inelastic hinges, i.e. under monotonic loading, there is a process of unloading in some of the hinges while the remaining hinges continue to increase their plastic rotations, damage and energy dissipation. In frames, as well as in continuum media, structural collapse is preceded by localization. Therefore the understanding

* Corresponding author. Address: Department of Structural Engineering, Faculated de Ingenieria, University of Los Andes, Mérida 5101, Venezuela. Tel./fax: +58-274-2402867.
E-mail address: marante@ula.ve (M.E. Marante).

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of this phenomenon is very important for the analysis of structures such as buildings or bridges subjected to earthquake or blast loadings.

Localization in framed structures presents an important difference with respect to the case of a continuum. In the latter, there has been identified only one localization mode: transfer of plastic strains to a narrow band with strain rate discontinuities (if time-independent plasticity is being considered). In the case of frames, the transfer of plastic rotations can occur in two different ways: transfer amongst the hinges around a node, as indicated in Fig. 1a, or transfer between the hinges in the same frame member (Fig. 1b). The procedures for the analysis of both kinds of localization effects are different. This paper only considers the latter case, i.e. localization in a frame member.

Localization in framed structures with plastic hinges has been studied in the past (see Maier et al., 1973; Bazant and Cedolin, 1991; Bazant and Kazemi, 1994; Bazant and Jirásek, 1996; Jirásek, 1997; amongst others). The present paper takes up the subject considered by Jirásek (1997). In that paper, the conditions for localization within a frame member with linear softening and without inelastic shortening were considered. Then, these results were used to study the postpeak behavior of some simple but representative structures. The present paper generalizes the results obtained in Jirásek (1997) by including general nonlinear softening and inelastic shortening effects in the frame member.

All these studies and the present paper deal with the problem of localization due to plasticity. Another related subject is the problem of localization due to buckling of structures (see for instance Pierre and Plaut, 1989; Zingales and Elishakoff, 2000). In that problem the analysis is carried out in a different framework, beam theory, and the nonlinearities that produce localization are of geometrical nature.

The procedure followed for localization analysis in the present paper differs from the one used in Jirásek (1997). In Nguyen and Bui (1974), a condition for the well-posedness of a quite general class of continuum plasticity models was proposed. This condition assures that a unique stress rate and internal variable rate are associated to any given strain rate. In Nguyen and Bui (1974) the problem of framed structures, plastic hinges or localization was not considered. However, it is shown in this paper that the general procedure of Nguyen and Bui, if adapted to the frame problem, leads to a simple, general and powerful method for the analysis of localization in this kind of structures.

The paper is organized as follows. In the second section, a model of the behavior of plastic hinges with softening is introduced. Three particular cases of plastic hinges within this general framework are presented. The first case corresponds to the linear softening hinge considered in Jirásek (1997). The second example corresponds to a plastic hinge with nonlinear softening, but still without axial shortening. The third example generalizes the linear model in Jirásek (1997) by the inclusion of the axial force effect.

In Section 3 of the paper, the equations that describe the behavior of a frame member with two plastic hinges are described. In Section 4, the condition for localization within a single frame member is derived.

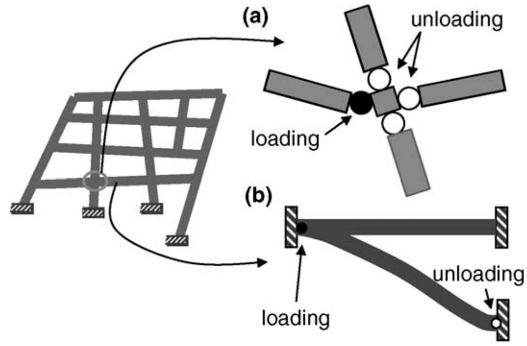

Fig. 1. (a) Localization across a node and (b) localization across a frame member.

First, the plastic model in the more general case is considered. Later, each one of the three particular cases of plastic hinges is developed as example.

In Section 5 of the paper, the differences between localization in a frame member and across a node are shown by the consideration of a simple framed structure and the numerical implications of the results obtained in this paper are discussed.

2. Plastic hinges with softening under bending and axial forces

2.1. General model

Let us consider a plastic hinge subjected to a bending moment m and an axial force n. Compression forces are assumed to be positive. The behavior of the plastic hinge is defined by a yield function f with hardening–softening terms h_1, h_2, etc. and a normality rule as follows:

$$f = f(m, n, h_1, h_2, \ldots); \qquad \dot{\phi}^{\mathrm{p}} = \lambda \frac{\partial f}{\partial m}; \qquad \dot{\delta}^{\mathrm{p}} = \lambda \frac{\partial f}{\partial n} \tag{1}$$

where $\dot{\phi}^{\mathrm{p}}$ is the plastic rotation rate, $\dot{\delta}^{\mathrm{p}}$ represents the permanent elongation rate and λ is the plastic multiplier of the hinge. The plastic work w of the hinge is defined by:

$$\dot{w} = m\dot{\phi}^{\mathrm{p}} + n\dot{\delta}^{\mathrm{p}} \tag{2}$$

The hardening terms are assumed to depend on the plastic work:

$$h_1 = h_1(w); \qquad h_2 = h_2(w); \ \ldots \tag{3}$$

In the following sections, the localization conditions for three particular cases will be derived as examples.

2.2. Plastic hinge with linear softening and without axial forces

The idealization of a softening hinge, without axial forces, proposed by Bazant and Kazemi (1994), and considered in Jirásek (1997), (see Fig. 2a) can be included in the precedent framework. The hinge presents linear softening as a function of the plastic rotation, although not as a function of the plastic work. The

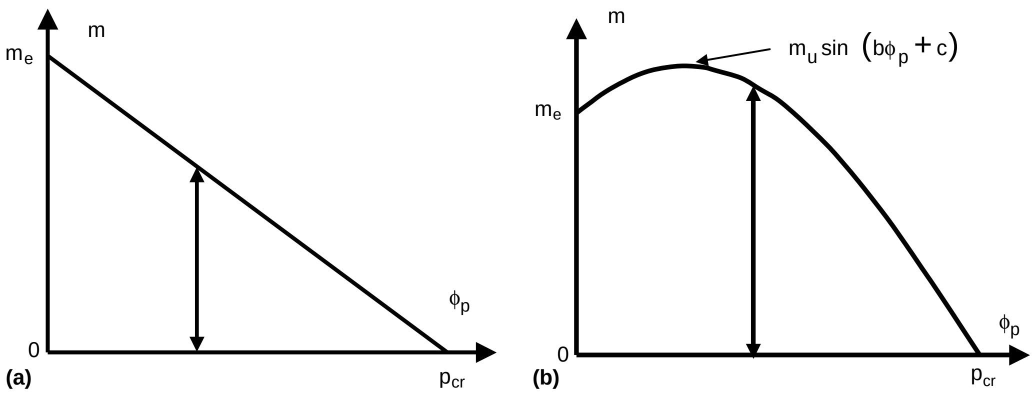

Fig. 2. (a) Plastic hinge with linear softening in plastic rotations and (b) plastic hinge with nonlinear softening.

behavior described in Fig. 2a is obtained by the following expressions for the yield function f and the hardening–softening term h:

$$f = |m| - (m_e + h); \quad h = \left(\frac{m_e}{p_{\text{cr}}}(p_{\text{cr}} m_e - 2w)\right)^{\frac{1}{2}} - m_e \tag{4}$$

2.3. Plastic hinge with sinusoidal softening and without axial force

The more realistic behavior shown in Fig. 2b can be represented by:

$$\begin{aligned} &f = |m| - (m_e + h); \quad h = (m_u^2 - (wb - \cos(c) m_u)^2)^{\frac{1}{2}} - m_e \\ &b = \frac{1}{p_{\text{cr}}}(\pi - c); \quad c = \arcsin\left(\frac{m_e}{m_u}\right) \end{aligned} \tag{5}$$

The behavior presented in Fig. 2b, describes the moment on the hinge as a sinusoidal function of the plastic rotation.

2.4. Plastic hinge with axial forces and linear softening in bending

In order to determine localization conditions under axial forces, the following expression of the yield function will be considered:

$$f = \left(\frac{m}{m_e + h}\right)^{\alpha} + \left(\frac{n}{n_e}\right)^{\beta} - 1 = 0 \tag{6}$$

where α, β, m_e and n_e are constants. The last parameter represents the yield axial force of the cross section. In the case of a rectangular cross section, $\alpha = 2/2$ and $\beta = 2$.

It can be noticed that only softening under bending is being considered. The hardening term h is again characterized by

$$h = \left(\frac{m_e}{p_{\text{cr}}}(p_{\text{cr}} m_e - 2w)\right)^{\frac{1}{2}} - m_e \tag{7}$$

so that, for zero axial forces, the model (6) and (7) describes a plastic hinge with linear softening as the one considered by Bazant and Kazemi (1994) and Jirásek (1997). The yield function and the plastic behavior of the hinge are represented in Fig. 3.

3. Elastic−plastic frame members with softening hinges

Let us consider a planar frame member between nodes i and j. The generalized displacements, or degrees of freedom, of the element are given by the matrix $\mathbf{u}^t = (u_i, v_i, \theta_i, u_j, v_j, \theta_j)$ (see Fig. 4a). This matrix and the generalized strain matrix $\mathbf{\Phi}^t = (\phi_i, \phi_j, \delta)$ (see Fig. 4b) define the kinematics of the frame member. It can be noticed that in a rigid body transformation of the frame member, the strain matrix is nil even if the displacement matrix is not. The relationship between displacements and strains is given by the following kinematic relation:

$$\dot{\mathbf{\Phi}} = \mathbf{B}\dot{\mathbf{u}}; \quad \mathbf{B} = \begin{bmatrix} \frac{\sin\alpha}{L} & -\frac{\cos\alpha}{L} & 1 & -\frac{\sin\alpha}{L} & \frac{\cos\alpha}{L} & 0 \\ \frac{\sin\alpha}{L} & -\frac{\cos\alpha}{L} & 0 & -\frac{\sin\alpha}{L} & \frac{\cos\alpha}{L} & 1 \\ \cos\alpha & \sin\alpha & 0 & -\cos\alpha & -\sin\alpha & 0 \end{bmatrix} \tag{8}$$

The variable conjugated to the strain matrix is the generalized stress matrix $\mathbf{M}^t = (m_i, m_j, n)$ (see Fig. 4c).

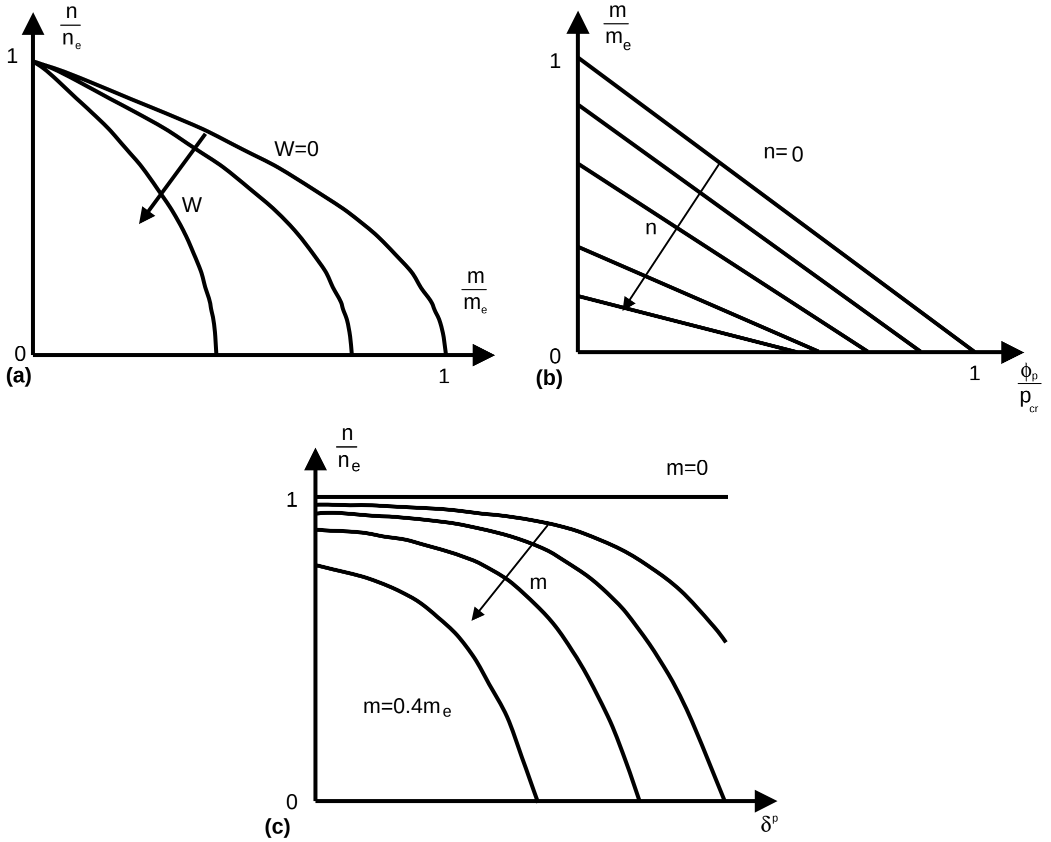

Fig. 3. (a) Yield function in the generalized stresses space ($\alpha = 2/2$ and $\beta = 2$), (b) moment as a function of the plastic rotation for different values of the axial force and (c) axial force as a function of the plastic elongation.

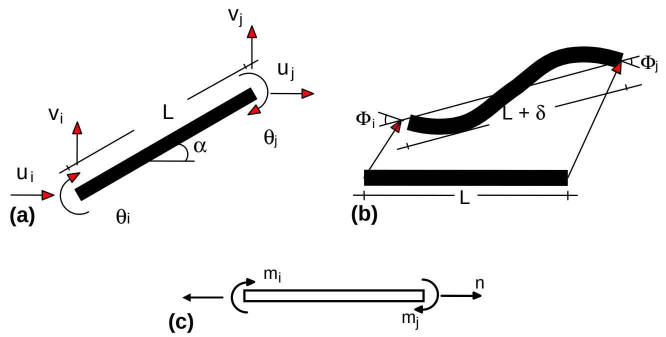

Fig. 4. (a) Generalized displacements of a frame member, (b) generalized deformations and (c) generalized stresses.

The behavior of the frame member is completely defined by the introduction of the generalized constitutive law that relates generalized stresses and strains. In the case of an elastic frame member, this relation is, of course, given by:

$$\mathbf{M} = \mathbf{S}\boldsymbol{\Phi} \tag{9}$$

where $\mathbf{S}$ is the local stiffness matrix of the frame member. In order to focus on bifurcations due to localization only, the geometrically nonlinear effects are neglected and the local stiffness matrix $\mathbf{S}$ has the conventional form:

$$\mathbf{S} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} & 0 \\ \frac{2EI}{L} & \frac{4EI}{L} & 0 \\ 0 & 0 & \frac{AE}{L} \end{bmatrix} \tag{10}$$

where E is elasticity modulus, A the area of the cross section, I is the inertia and L the length of the frame member.

In the case of an elastic–plastic frame member, the conventional lumped plasticity representation is adopted (see Fig. 5). The plastic strain matrix of a frame member is now introduced as: $\mathbf{\Phi}_p^t = (\phi_i^p, \phi_j^p, \delta^p)$ where ϕ_i^p and ϕ_j^p are, respectively, the plastic rotations of plastic hinge i and plastic hinge j. δ^p is the permanent elongation of the chord that results from adding the respective values at the plastic hinges i and j ($\delta^p = \delta_i^p + \delta_j^p$).

An additive decomposition of the total strains into an elastic part (generalized strains of the elastic beam-column) and the plastic part (generalized strains of the plastic hinges) is assumed:

$$\mathbf{\Phi} = \mathbf{\Phi}^e + \mathbf{\Phi}^p \tag{11}$$

Taking into account that the elastic strains obey the elasticity law (9), the state law of an elastic–plastic frame member can be written as:

$$\mathbf{M} = \mathbf{S}(\mathbf{\Phi} - \mathbf{\Phi}^p) \tag{12}$$

The behavior of the hinges i and j is defined by the yield functions $f_i = f_i(m_i, n, w_i) \leqslant 0$ and $f_j = f_j(m_j, n, w_j) \leqslant 0$. Where the terms w_i and w_j represent the plastic work of, respectively, hinges i and j. The normality rule for both hinges leads to:

$$\dot{\phi}_i^p = \lambda_i \frac{\partial f_i}{\partial m_i}; \qquad \dot{\phi}_j^p = \lambda_j \frac{\partial f_j}{\partial m_j}; \qquad \dot{\delta}_p = \lambda_i \frac{\partial f_i}{\partial n} + \lambda_j \frac{\partial f_j}{\partial n} \tag{13}$$

where λ_i and λ_j are the plastic multipliers of the hinges. The relationship between plastic work and plastic multipliers can be written as:

$$\dot{w}_i = \lambda_i l_i(m_i, n, w_i); \qquad \dot{w}_j = \lambda_j l_j(m_j, n, w_j) \tag{14}$$

where

$$l_i = m_i \frac{\partial f_i}{\partial m_i} + n \frac{\partial f_i}{\partial n}; \qquad l_j = m_j \frac{\partial f_j}{\partial m_j} + n \frac{\partial f_j}{\partial n} \tag{15}$$

The set of equations (8) and (12)–(15) defines a finite element that can be included in the library of any conventional structural analysis program.

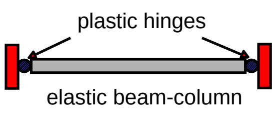

Fig. 5. Lumped plasticity model of a frame member.

4. Analysis of the uniqueness of the member response by the method of Nguyen and Bui

4.1. Localization in the general case

In this section, the uniqueness of the solution of the problem defined in Section 3 will be analyzed. The uniqueness conditions can be obtained from the equations that express the plastic multipliers as a function of the strain rate or the displacement rate. The procedure to derive these equations is now described.

The rate of generalized stresses is given by:

$$\dot{\mathbf{M}} = \mathbf{S}\dot{\mathbf{\Phi}} - \mathbf{S}\dot{\mathbf{\Phi}}^{\mathrm{p}} \tag{16}$$

The normality rule (13), the evolution law of the plastic work (14) and the consistence conditions $\dot{f}_i \leqslant 0$ and $\dot{f}_j \leqslant 0$, are written in matrix notation as follows:

$$\dot{\mathbf{\Phi}}^{\mathrm{p}} = \frac{\partial \mathbf{F}}{\partial \mathbf{M}}^{\mathrm{t}} \mathbf{\Lambda} \quad \text{where } \frac{\partial \mathbf{F}}{\partial \mathbf{M}} = \begin{bmatrix} \frac{\partial f_i}{\partial m_i} & 0 & \frac{\partial f_i}{\partial n} \\ 0 & \frac{\partial f_j}{\partial m_j} & \frac{\partial f_j}{\partial n} \end{bmatrix}; \ \mathbf{\Lambda} = \begin{bmatrix} \lambda_i \\ \lambda_j \end{bmatrix} \tag{17}$$

$$\dot{\mathbf{W}} = \mathbf{L}\mathbf{\Lambda} \quad \text{where } \mathbf{W} = \begin{bmatrix} w_i \\ w_j \end{bmatrix}; \ \mathbf{L} = \begin{bmatrix} l_i & 0 \\ 0 & l_j \end{bmatrix} \tag{18}$$

$$\dot{\mathbf{F}} = \frac{\partial \mathbf{F}}{\partial \mathbf{M}} \dot{\mathbf{M}} + \frac{\partial \mathbf{F}}{\partial \mathbf{W}} \dot{\mathbf{W}} \leqslant 0 \quad \text{where } \mathbf{F} = \begin{bmatrix} f_i \\ f_j \end{bmatrix}; \ \frac{\partial \mathbf{F}}{\partial \mathbf{W}} = \begin{bmatrix} \frac{\partial f_i}{\partial w_i} & 0 \\ 0 & \frac{\partial f_j}{\partial w_j} \end{bmatrix} \tag{19}$$

The combination of (17)–(19) leads to the following inequality:

$$\mathbf{D} \leqslant \mathbf{C}\Lambda \quad \text{where } \mathbf{D} = \frac{\partial \mathbf{F}}{\partial \mathbf{M}} \mathbf{S}\dot{\Phi}; \ \mathbf{C} = \frac{\partial \mathbf{F}}{\partial \mathbf{M}} \mathbf{S} \frac{\partial \mathbf{F}}{\partial \mathbf{M}}^{\mathrm{t}} - \frac{\partial \mathbf{F}}{\partial \mathbf{W}} \mathbf{L} \tag{20}$$

Thus, the problem of the computation of the plastic multipliers as a function of the total strain rate and the state of the member is defined as follows:

$$\begin{cases} \text{if } \lambda_i > 0 & C_{11}\lambda_i + C_{12}\lambda_j = D_1 \\ \text{if } \lambda_i = 0 & C_{12}\lambda_j \geqslant D_1 \end{cases}; \qquad \begin{cases} \text{if } \lambda_j > 0 & C_{21}\lambda_i + C_{22}\lambda_j = D_2 \\ \text{if } \lambda_j = 0 & C_{21}\lambda_i \geqslant D_2 \end{cases} \tag{21}$$

It can be noticed that (21) corresponds to the Euler conditions of the following minimization problem if the matrix **C** is symmetric (see Lions, 1968):

$$\min J = \tfrac{1}{2}\Lambda^{\mathrm{t}}\mathbf{C}\Lambda - \Lambda^{\mathrm{t}}\mathbf{D}; \quad \Lambda \geqslant 0 \tag{22}$$

This problem has a unique solution if the matrix **C** is positive definite (J is then a convex function, Lions, 1968), i.e. if its eigenvalues are positive. Thus, the uniqueness condition for the problem (22) can be written as:

$$C_{\mathrm{II}} > 0 \tag{23}$$

where C_{II} is the smallest eigenvalue of the matrix **C**. If C_{II} is negative or nil, there can be more than one solution for the plastic multipliers for a given set of strain rates. Thus, the surface that divides the zone of uniqueness from the localization zone is given by:

$$C_{\mathrm{II}} = 0 \tag{24}$$

It can be noticed that the following equation is valid at the surface (24):

$$\mathrm{Det}(\mathbf{C}) = 0 \tag{25}$$

which reminds the localization condition of a continuum.

Alternatively, the plastic multipliers can be expressed as a function of the generalized displacements by using the kinematic equation (8). Then, the inequality (20) becomes:

$$\mathbf{E} \leqslant \mathbf{C}\Lambda \quad \text{where } \mathbf{E} = \frac{\partial \mathbf{F}}{\partial \mathbf{M}}\mathbf{S}\mathbf{B}\dot{\mathbf{u}}; \ \mathbf{C} = \frac{\partial \mathbf{F}}{\partial \mathbf{M}}\mathbf{S}\frac{\partial \mathbf{F}}{\partial \mathbf{M}}^{t} - \frac{\partial \mathbf{F}}{\partial \mathbf{W}}\mathbf{L} \tag{26}$$

which, of course, does not change the uniqueness condition (23).

4.2. Localization in a frame member with linear softening and without axial forces

Let us consider the model defined by the yield function (4). Then, the matrix $\mathbf{C}$ becomes:

$$\mathbf{C} = \begin{bmatrix} \dfrac{4EI}{L} + m_i \dfrac{\mathrm{d}h(w_i)}{\mathrm{d}w_i} & \dfrac{2EI}{L}\dfrac{\partial f_i}{\partial m_i}\dfrac{\partial f_j}{\partial m_j} \\ \dfrac{2EI}{L}\dfrac{\partial f_i}{\partial m_i}\dfrac{\partial f_j}{\partial m_j} & \dfrac{4EI}{L} + m_j \dfrac{\mathrm{d}h(w_j)}{\mathrm{d}w_j} \end{bmatrix} \tag{27}$$

The localization conditions when both plastic hinges are active will be considered. Thus, the moment m_i and m_j can be expressed as a function of the plastic work from the equations $f_i = 0$ and $f_j = 0$.

$$|m_i| = \frac{\sqrt{p_{\mathrm{cr}} m_e (p_{\mathrm{cr}} m_e - 2w_i)}}{p_{\mathrm{cr}}}; \qquad |m_j| = \frac{\sqrt{p_{\mathrm{cr}} m_e (p_{\mathrm{cr}} m_e - 2w_j)}}{p_{\mathrm{cr}}} \tag{28}$$

The computation of the terms $\frac{\mathrm{d}h(w_i)}{\mathrm{d}w_i}$ and $\frac{\mathrm{d}h(w_j)}{\mathrm{d}w_j}$ from the hardening function (4) and the evaluation of the smallest eigenvalue of matrix $\mathbf{C}$ lead to the following uniqueness condition:

$$C_{\mathrm{II}} = \frac{2EI}{L} - \frac{m_y}{p_{\mathrm{cr}}} > 0 \tag{29}$$

This is, of course, the result presented in Jirásek (1997). If C_{II} is equal to zero or negative, there may be three different solutions for the problem (20). They are characterized by $\lambda_i > 0$, $\lambda_j > 0$ (softening in both hinges), $\lambda_i = 0$, $\lambda_j > 0$ (unloading in hinge i and softening in hinge j) and $\lambda_i > 0$, $\lambda_j = 0$ (softening in hinge i and unloading in hinge j). The result (29) can be interpreted as follows: if the length of the frame member exceeds the critical value given by (29) there will appear bifurcated solutions with localization at the frame member level.

4.3. Localization in the case of nonlinear isotropic softening without inelastic shortening

For any hardening function h, the smallest eigenvalue of the matrix $\mathbf{C}$, as defined by (27), is given by:

$$C_{\mathrm{II}} = \frac{4EI}{L} + \frac{Q_i + Q_j}{2} - \sqrt{\left(\frac{Q_i - Q_j}{2}\right)^2 + \left(\frac{2EI}{L}\right)^2} \tag{30}$$

where $Q_i = m_i \frac{\mathrm{d}h(w_i)}{\mathrm{d}w_i}$ and $Q_j = m_j \frac{\mathrm{d}h(w_j)}{\mathrm{d}w_j}$. The domain of localization characterized by $C_{\mathrm{II}} < 0$ is represented in Fig. 6.

It can be noticed that the curve $C_{\mathrm{II}} = 0$ presents two asymptotes given by: $Q_i = -\frac{4EI}{L}$ and $Q_j = -\frac{4EI}{L}$. The intersection of the curve $C_{\mathrm{II}} = 0$ with the line $Q_i = Q_j$ occurs at the point $(-\frac{2EI}{L}, -\frac{2EI}{L})$ which results again in the Jirásek uniqueness condition.

The localization domain for the particular case of sinusoidal softening is shown in Fig. 7. The three curves that appear in the figure correspond to the condition $C_{\mathrm{II}} = 0$ for different relationships between EI/L and m_u/p_{cr} (specifically 0.3, 0.5 and 0.7). The curves are represented in the space of plastic work of both hinges. In this case, no localization is possible for nil plastic work since the hinge experiences first a process

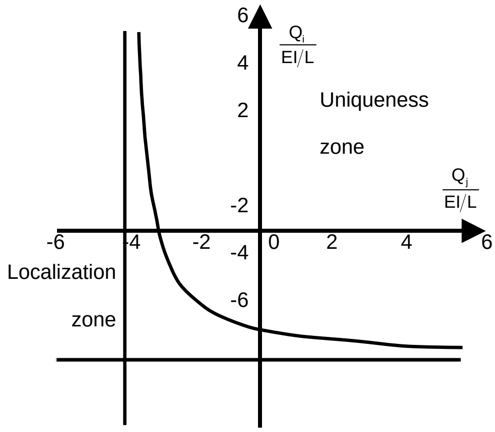

Fig. 6. Localization domain with no axial forces.

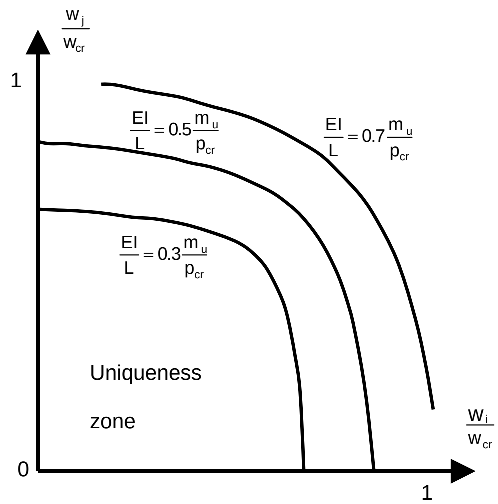

Fig. 7. Localization domain for sinusoidal hardening.

of hardening. The uniqueness zone is then included between the origin and the corresponding curve. It can be noticed that the uniqueness domain reduces its size when the relation between EI/L and m_u/p_{cr} decreases.

4.4. Localization in a frame member with axial forces and linear softening in bending

The last example of this section corresponds to a frame member with inelastic shortening and plastic rotations as described by the yield function (6). As aforementioned, this expression leads to the linear softening case when the axial forces are nil. Therefore, there may be localization when both hinges are active ($f_i = 0$ and $f_j = 0$) but there is still no plastic dissipation ($w_i = 0$ and $w_j = 0$). This is the case that will be considered in this section. Additionally, if a rectangular cross section is analyzed ($\alpha = 2/2$ and $\beta = 2$), the expression of C_{II} is given by:

$$C_{\text{II}} = \frac{2p_{\text{cr}}n_e^4\frac{EI}{L} - m_e n_e^4 + m_e n^4}{p_{\text{cr}}m_e^2 n_e^4} \tag{31}$$

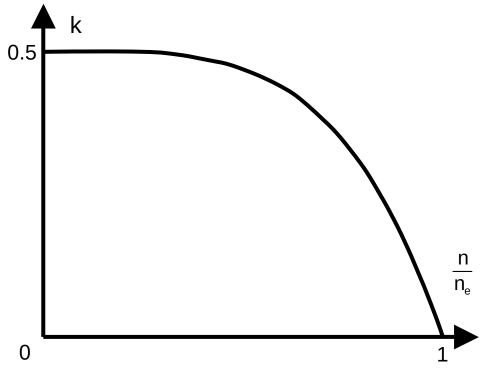

Fig. 8. Variation of the factor $k(n)$.

and the inequality $C_{II} > 0$ leads to the following relationship between EI/L and m_e/p_{cr}:

$$\frac{EI}{L} > k(n)\frac{m_e}{p_{cr}}; \quad k(n) = \frac{1}{2}\frac{(n_e^4 - n^4)}{n_e^4} \tag{32}$$

It can be noticed that for $n = 0$, the criterion $C_{II} > 0$ results again in (29). The graph of $k(n)$ can be seen in Fig. 8.

If the axial force tends to n_e, localization can occur for increasingly smaller values of the moment on the hinge. It is also noted that the localization criterion does not depend on the axial stiffness of the member. This is due to the fact that there is no hardening function associated to the axial forces in the yielding function (7). As a result, a particularly simple expression is obtained. This is not always the case. If full isotropic hardening is considered, the localization criterion depends on both stiffness parameters and it is very difficult to evaluate qualitatively the corresponding analytical expression. A numerical study of each particular model is then needed.

5. Localization in framed structures

5.1. Equilibrium equations

The strain–displacement relationship (8) and the generalized constitutive laws (12)–(15) describe only the behavior of a frame member. In order to carry out a localization analysis of a framed structure, the introduction of an equilibrium equation is also required. The internal or deformation power of a frame member is given by:

$$P_i^* = \mathbf{M}^t\dot{\mathbf{\Phi}}^* \tag{33}$$

The external power is defined by the introduction of the matrix of external forces $\mathbf{P}$, then:

$$P_e^* = \mathbf{P}^t\dot{\mathbf{U}}^* \tag{34}$$

where the matrix $\mathbf{U}$ has the nodal displacements of the entire frame. Thus, the principle of virtual power implies:

$$\sum_{b=1}^{m}\mathbf{M}^t\dot{\mathbf{\Phi}}^* = \left(\sum_{b=1}^{m}\mathbf{M}^t\mathbf{B}\right)\dot{\mathbf{U}}^* = \mathbf{P}^t\dot{\mathbf{U}}^* \quad \forall\dot{\mathbf{U}}^*; \text{ i.e. } \sum_{b=1}^{m}\mathbf{B}^t\mathbf{M} = \mathbf{P} \tag{35}$$

It can be noticed that with this notation, the analysis of framed structures has the same form of the mechanics of a continuum. That is, in terms of "stresses", "strains", "strain–displacement relationship", "constitutive laws" and "equilibrium equations". In this way, the similarities and differences between localization in a continuum and in a framed structure can be more easily appreciated.

5.2. Localization analysis in a simple frame

In this section, a very simple structure will be considered (see Fig. 9). In some aspects, this example may be representative of what happens in a node of a frame under lateral forces.

A simply supported beam is subjected to vertical displacements v in the column. The structure will be modeled using the frame elements A and B between, respectively, the nodes 1, 2 and 3, 4. The equilibrium equations (35) in terms of the stress rates give:

$$\dot{m}_1 = 0; \qquad \dot{m}_4 = 0; \qquad \dot{m}_2 + \dot{m}_3 = 0; \qquad \dot{m}_3 - \dot{m}_2 = \dot{P}L \tag{36}$$

the strain–displacement relationship (8) for this problem leads to:

$$\dot{\phi}_2 = \frac{\dot{v}}{L} + \dot{\theta}; \qquad \dot{\phi}_3 = -\frac{\dot{v}}{L} + \dot{\theta} \tag{37}$$

where θ is the rotation of the column. The constitutive law in terms of stress and strain rates can be written as:

$$\begin{matrix} \dot{\phi}_2 = F_{12}\dot{m}_1 + F_{22}\dot{m}_2 \\ \dot{\phi}_3 = F_{33}\dot{m}_3 + F_{34}\dot{m}_4 \end{matrix}; \quad \text{i.e.} \quad \begin{matrix} \dot{m}_2 = K_A\dot{\phi}_2 \\ \dot{m}_3 = K_B\dot{\phi}_3 \end{matrix} \quad \text{where } K_A = 1/F_{22} \text{ and } K_B = 1/F_{33} \tag{38}$$

where the explicit expression of F_{12}, F_{22}, F_{33}, F_{34}, and K_A and K_B can be obtained from the elastic stiffness matrix **S** and the yield functions defined in Sections 2 and 3. A representation of the constitutive laws (38) can be seen in Fig. 10. Additionally a graphical interpretation of the equilibrium equation $\dot{m}_2 + \dot{m}_3 = 0$ can be seen in the same figure. It can be noticed that this expression imposes the same moment (in absolute value) on both sides of the column. Therefore, if the state of the frame members A and B are represented by the points (m_2, ϕ_2) and $(-m_3, -\phi_3)$ in the graphs of Fig. 10 then both points must lie on the same horizontal line.

It can be shown, by algebraic manipulation of the expressions (36)–(38), that the rotation θ and force P rates are given by:

$$\begin{aligned} (K_A - K_B)\frac{\dot{v}}{L} + (K_A + K_B)\dot{\theta} = 0 \\ (K_A + K_B)\frac{\dot{v}}{L} + (K_A - K_B)\dot{\theta} = -\dot{P}L \end{aligned} \tag{39}$$

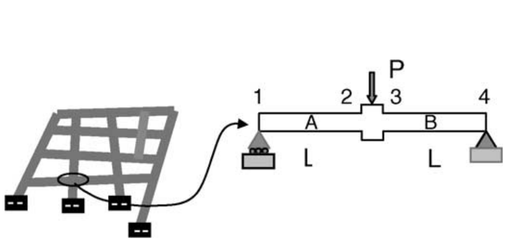

Fig. 9. A simple frame as an example of the behavior of a joint in a large structure.

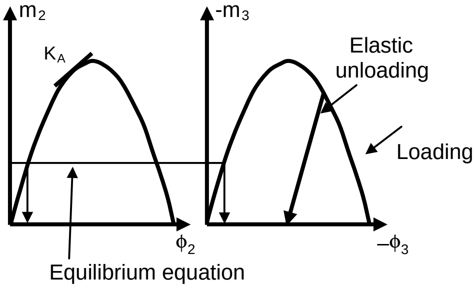

Fig. 10. Constitutive laws for the two-element frame.

Assume that the points (m_2, ϕ_2) and $(-m_3, -\phi_3)$ are before the peak of the curves in Fig. 10. Then, it is clear that there is only one solution that satisfies the constitutive laws and the equilibrium equations at the same time. This solution is given by:

$$K_A = K_B > 0; \qquad \dot{\theta} = 0; \qquad \dot{P} = -(K_A + K_B)\frac{\dot{v}}{L^2}; \tag{40}$$

However, if the points are after the peak (see Fig. 11a), it is clear that there are three possible solutions: the one represented in (40) and two others with localization in the frame element A or in the frame element B. Assuming localization in the element A (see Fig. 11) this solution is given by:

$$K_A \leqslant 0; \qquad K_B > 0; \qquad \dot{\theta} = \frac{K_B - K_A}{(K_A + K_B)L}\dot{v}; \qquad \dot{P} = \frac{-4K_B K_A}{(K_A + K_B)L^2}\dot{v} \tag{41}$$

It can be noticed that the requirements for this solution are $K_A \leqslant 0$ and $K_A + K_B \neq 0$. Other simple structures can be analyzed with the same qualitative results of this example.

5.3. Localization in a node and localization in a frame member

In the previous example, localization is possible even if the uniqueness condition (23) is verified. The explanation of this apparent contradiction is that the criterion discussed in Section 4 is a condition of localization within a frame member (the transfer of energy dissipation occurred through a frame member) and the localization presented in the previous example occurred in a node (the transfer of energy dissipation occurred through a node).

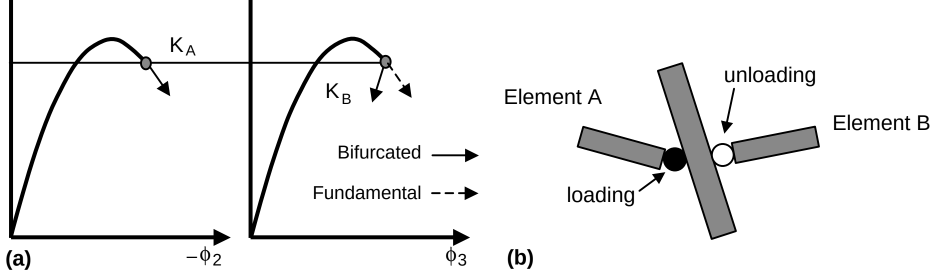

Fig. 11. (a) Bifurcated and fundamental solutions and (b) localization in hinge A.

For the analysis of localization in a frame member only the constitutive equations are needed; in the case of localization in a node, equilibrium as well as kinematic equations are also required. Therefore, the analysis of localization in a node is formally similar to the case of localization in a continuum. Remember that in a continuum, the equations for localization analysis are the Maxwell compatibility conditions (a kinematic relationship), the equilibrium equation across the discontinuity surface and the constitutive laws. There is no equivalent of localization in a frame member using the continuum framework. The reason is that, usually, it is assumed that the constitutive law must always define a unique relationship between stresses and the history of strains. There is no physical support for this kind of assumption in the case of a framed structure. Thus a second class of localization is possible in frames and has been the subject of this paper.

5.4. Analysis of localization in a node

A general criterion of localization in a node is not yet available in the literature. Only particular cases as the one presented in Section 5.2 have been treated. However, global criteria of uniqueness and stability for elastic and elastic–plastic structures as well as damage mechanics do exist in the literature (see for instance Koiter, 1945; Budiansky, 1974; Hill, 1978; Cen and Maier, 1992; Bolzon et al., 1997; Nguyen, 2000; Cocchetti et al., 2002). As a first approximation, the analysis of localization in a node of a frame with plastic hinges and softening could be carried out by the consideration of an equivalent nonlinear elastic structure in the loading mode (this is what is done in a continuum). If a bifurcation is detected by the consideration of the global problem and the localization criterion (23) is not detected, then localization can only occur in a node since bifurcations of geometrical nature are not possible in the framework considered in this paper. However this is not a completely satisfying approach and a deeper analysis of the localization problem in a node is important.

A question that naturally arises is if the global criteria of bifurcation include as a particular case the local criterion of localization at the frame level. The answer is no if the global criterion is based on the existence of a unique relationship between the strain and the stress rates. For instance the analysis carried out in Section 5.2 was based on the assumption that such a unique relationship does exist (expressions (38)). However, it has been shown in Section 5 that if the localization criterion is verified, there is more than one stress rate that correspond to a given strain rate. Therefore, the validity of the global criteria of bifurcation remains to be proven when localization at the frame member level is possible. The authors of this paper do not know of global bifurcation analyses when the constitutive laws do not assure a unique relationship between stresses and the history of strains. In the absence of global criteria of bifurcation based directly on the nonlinear constitutive laws instead of the stress and strain rates problem, the only alternative is the analysis of the frame at both levels: frame member (local level) and the structure (global level).

5.5. Numerical implications of the localization criterion in a frame member

The numerical analysis of framed structures is usually carried out by a conventional step by step procedure. The time interval $[0, T]$ of the analysis is then substituted by a discrete set of times $(0, t_1, t_2, \ldots, T)$. The analysis at some time t_a can be performed if the state of the structure at the time t_{a-1} is known. In that case, the constitutive law and the strain–displacement relationship presented in Section 3 describe a relation between the stress and the displacement matrices:

$$\mathbf{M} = \mathbf{M}(\mathbf{U}) \tag{42}$$

Eqs. (35) and (42) lead to:

$$\mathbf{L}(\mathbf{U}) = \sum_{b=1}^{m} \mathbf{B}^{t}\mathbf{M}(\mathbf{U}) - \mathbf{P} = \mathbf{0} \tag{43}$$

The resolution of (43) is called "global problem" and is carried out by the Newton method. Thus for each iteration of the local problem, the computation of the stresses **M** as a function of the displacements **U** is needed. This computation is denoted "local problem" and in the general case is a nonlinear problem too.

It can be noticed that the localization criterion proposed in Section 4, represents a mathematical analysis of the local problem. Thus, a complete bifurcation analysis during the numerical resolution of framed structures should include the study of the global problem by the methods mentioned in the previous section and the analysis of each local problem by the criterion developed in Section 4.

In a finite element analysis of a continuum, the local problem is solved for each Gauss point of the finite element mesh. In that case, the localization analysis at the integration point is not needed since the constitutive law is assumed to rule out that possibility. This is not the case of framed structures.

6. Summary and conclusions

It has been shown that the Nguyen and Bui method allows for the determination of a completely general localization criterion within frame members with plastic hinges, i.e. at the local, frame member level. Analytical results in the more complicated cases can be obtained with the use of any symbolic manipulator program, although the results may become unreadable very quickly. In fact, the same method can be used for the analysis of more general cases than the one considered in this paper, for instance a fully three-dimensional frame member. This was not the case of this paper because the authors thought that the fundamental issues could be presented more clearly for two-dimensional frame members.

It is important to underline that this is a localization criterion within a frame member, localization can also happen at a node of the frame, i.e. there may be localization in the hinge of one of the members connected to a node while the remaining members unload. This very important case was only marginally considered in this paper.

By comparison with the continuum mechanics case, it can be noticed that localization in framed structures exhibits a significant difference. There is one localization mode in the former case; there are two distinct modes in the latter. The possible interactions between both modes have not been analyzed in this paper and remain an open problem.

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